EINSTEIN SU(3) AND G_2 STRUCTURES

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Abstract. We describe a method to obtain SU(3)-structures and G_2-structures on 6 and 7-dimensional manifolds respectively, such that its associated metric is Einstein. More concretely, we have that different classes of SU(2) and SU(3)-structures, on 5 and 6-dimensional manifolds whose induced metric is Einstein can produce, via warped products, different classes SU(3) and G_2-structures such that its associated metric is also Einstein.

Introduction

The relation between some geometric structures (like Hermitian, SU(3) or G_2-structures) and Einstein metrics has been deeply studied by many different authors. In particular one of the most important problems related with this issue is the longstanding conjecture due to Goldberg [21]:

"*A compact almost Kähler Einstein manifold is Kähler*".

Partial affirmative answers have been obtained under some additional curvature conditions. For example in [24] Sekigawa proved that assuming non-negative scalar curvature the conjecture is true. However, the general case is still open. Concerning the non-compact version of this conjecture; Apostolov, Draghici and Moroianu found a counterexample which is described in [1]. This example consists on a non-compact solvable Lie group endowed with a left-invariant almost Kähler structure whose induced metric is Einstein. Can be easily checked that the almost complex structure is not complex since its Nijenhuis tensor does not vanish and thus, the almost Kähler structure is not Kähler.

Concerning the relation between SU(3)-manifolds and Einstein metrics, in [3] Bedulli and Vezzoni give a detailed description of the Ricci tensor of a SU(3)-manifold in terms of the torsion forms. In section 2 we give some results describing how to obtain different classes of Einstein SU(3)-manifolds via the warped product of an Einstein 5-dimensional SU(2)-manifold.

Section 3 is focused on G_2-manifolds. A 7-dimensional manifold is called a G_2-manifold if it admits a G_2-structure or equivalently, if its structure group reduces to the exceptional Lie group G_2. The existence of a G_2-structure on a manifold can also be characterized by the presence of a three-form on the manifold (usually called the fundamental form) which is globally defined and non-degenerate.

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In [16], the authors distinguish 16 classes of G$_2$-structures attending to the decomposition of the covariant derivative of the fundamental three-form into G$_2$-irreducible components. Equivalently this classification can be described in terms of the behavior of the derivatives of the fundamental three-form. We will mainly focus our attention on closed and coclosed G$_2$-structures (also called calibrated and cocalibrated), that is, G$_2$-manifolds such that its fundamental form is closed or coclosed, respectively. In particular a G$_2$-structure which fundamental form is closed and coclosed simultaneously is called parallel -because the fundamental form is parallel with respect to the Levi-Civita connection- and its Riemannian holonomy group is a subgroup of the exceptional Lie group G$_2$. These manifolds are Ricci-flat.

Gibbons, Page and Pope described a G$_2$-analogue of the Goldberg conjecture in [22] were they study supersymmetric string solutions on closed G$_2$-manifolds. This analogue can be stated as follows:

"*A compact Einstein calibrated* G$_2$ *manifold is Parallel*".

In [8] Cleyton and Ivanov answer positively to this question. For the non-compact version several authors have given partial affirmative answers under some additional conditions. For example, in [4] the author shows that every Einstein closed G$_2$ manifold with non-negative scalar curvature is Parallel. In [9] the authors proved that Einstein closed G$_2$-manifolds which are also $*$-Einstein are, in fact, Parallel. In [15] it is shown that in contrast to the almost Kähler case, a seven-dimensional solvmanifold cannot admit any left invariant calibrated G$_2$-structure such that its induced metric is Einstein, unless it is Parallel.

Concerning the analogous problem asking the G$_2$-structure to be coclosed, that is

"*A compact Einstein cocalibrated* G$_2$ *manifold is Parallel*",

it is well-known that the answer is no since Nearly Parallel and 3-Sasakian manifolds are counterexamples for this statement.

Along section 3 we describe different results that allow us to obtain Einstein manifolds endowed with different classes of G$_2$-structures via warped products. Finally, using Proposition 3.3 we describe an Einstein cocalibrated (non-Ricci flat) G$_2$ manifold which is not Nearly Parallel neither a 3-Sasakian structure.

1. Warped Products

Let (B, g_B) and (F, g_F) be two Riemannian manifolds, and let $f > 0$ be a real differentiable function on B. We denote by π and σ the projections of $B \times F$ onto B and F, respectively. The warped product, namely $M = B \times_f F$, is the product manifold $M = B \times F$ endowed with the metric g given by

$$g = \pi^*(g_B) + f^2\sigma^*(g_F).$$

The manifold B is called the base of M, F the fibre, and the warped product is called trivial if f is a constant function. We denote by Ric^B the lift (pullback by π)

of the Ricci curvature of B, similarly for Ric^F; and let $Hess(f)$ be the lift to M of the Hessian of f. By [23, p. 211] the warped product (M, g), where $M = B \times_f F$, is Einstein with $Ric = \lambda g$ if and only if (F, g_F) is Einstein ($Ric^F = \mu g_F$), with Einstein constant μ, and the following conditions are satisfied

$$\lambda g_B = Ric^B - \tfrac{d}{f} Hess(f),$$
$$\lambda = \tfrac{\mu}{f^2} - \tfrac{\Delta f}{f} - (d-1)\left|\tfrac{\nabla f}{f}\right|^2_{g_B},$$

where $\Delta f = tr\big(Hess(f)\big)$, ∇f denotes the gradient of f and $d = \dim(F)$.

Moreover, when the base space has dimension 1 ($\dim(B) = 1$), then these equations reduce to

$$(f')^2 + \frac{\lambda}{d} f^2 = \frac{\mu}{d-1}. \tag{1}$$

The behavior of the solutions of (1) depend on the signs of λ and μ. Nevertheless, up to homotheties, those solutions (besides the constant case) are given in the following table (Table 1) (see also [2]).

Table 1. **Solutions of the system** (1)

μ	$-(d-1)$	0	$d-1$	$d-1$	$d-1$
λ	$-d$	$-d$	$-d$	0	d
f	$\cosh t$	e^t	$\sinh t$	t	$\sin t$

From this table follows the next result.

Theorem 1.1 (Theorem 9.110, [2])**.** *Let (M, g) be a warped product, where $M = B \times_f F$, $\dim(B) = 1$ and $\dim(F) = d > 1$. If (M, g) is a complete Einstein manifold, then either M is a Ricci-flat Riemannian product, or $B = \mathbb{R}$, F is Einsten with non-positive scalar curvature and M has negative scalar curvature.*

2. Warped product of SU(2)-structures

This section is devoted to the construction of Einstein SU(3)-structures on the manifold $I \times L$ from Einstein SU(2)-structures on L.

It is well known that SU(2)-structures and SU(3)-structures are closely related. In particular if $(L, \eta, \omega_1, \omega_2, \omega_3)$ is a 5-dimensional manifold endowed with a SU(2)-structure then the forms

$$\omega = \omega_1 + \eta \wedge ds, \qquad \Psi = \psi_+ + i\,\psi_- = (\omega_2 + i\,\omega_3) \wedge (\eta + i\,ds), \tag{2}$$

define an SU(3)-structure on the 6-dimensional manifold $\mathbb{R} \times L$ where s denotes the coordinate in $\mathbb{R}$.

Conversely if L is an oriented hypersurface of a SU(3) manifold $(N, \omega, \psi_+, \psi_-)$ then the forms on L defined by

$$\eta = \iota_V \omega, \quad \omega_1 = \pi^* \omega, \quad \omega_2 = \iota_V \psi_-, \quad \textit{and} \quad \omega_3 = -\iota_V \psi_+$$

with V the unitary vector field of N normal to L and π the projection of N onto L, describe a SU(2)-structure on L.

Anyway, the aim of this section is to construct SU(3)-structures from SU(2)-structures in such a way that the Einstein condition of the induced metric is preserved along this construction. Thus, let $(L, \eta, \omega_1, \omega_2, \omega_3)$ be an SU(2) manifold. Consider a family of SU(2)-structures $(\eta(s), \omega_1(s), \omega_2(s), \omega_3(s))$ on L, for any s in some interval I, such that

$$\boxed{\begin{aligned} \eta(s) &= f(s)\,\eta, \\ \omega_1(s) &= f^2(s)\,\omega_1, \\ \omega_2(s) &= f^2(s)\,\omega_2, \\ \omega_3(s) &= f^2(s)\,\omega_3, \end{aligned}}$$

where $f = f(s) > 0$ is a real differentiable function on I. As a consequence of (2), the differential forms $(\omega(s), \psi_+(s), \psi_-(s))$ given by

$$\begin{aligned} \omega(s) &= \omega_1(s) + \eta(s) \wedge ds, \\ \psi_+(s) &= \omega_2(s) \wedge \eta(s) - \omega_3(s) \wedge ds, \\ \psi_-(s) &= \omega_3(s) \wedge \eta(s) + \omega_2(s) \wedge ds, \end{aligned}$$

describe a SU(3)-structure on $I \times L$ for any $s \in I$. Notice that the forms $\omega(s), \psi_+(s)$ and $\psi_-(s)$ can be expressed in terms of the function $f(s)$ and the SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on L, as follows

$$\begin{aligned} \omega(s) &= f^2(s)\omega_1 + f(s)\eta \wedge ds, \\ \psi_+(s) &= f^3(s)\omega_2 \wedge \eta - f^2(s)\omega_3 \wedge ds, \\ \psi_-(s) &= f^2(s)\omega_3 \wedge \eta + f^2(s)\omega_2 \wedge ds. \end{aligned} \tag{3}$$

Thus, one can check that the metric induced by the SU(3)-structure, namely g_N is exactly the warped product metric

$$g_N = ds^2 + f^2(s)\, g_L \tag{4}$$

on $I \times_f L$, where g_L denotes the metric induced by the SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on L.

If we denote by $\hat{d}$ the differential on $N = I \times L$ and by $\tilde{d}$ the differential on L, from (3) we have that

$$\begin{aligned} \hat{d}\omega(s) &= f^2(s)\tilde{d}\omega_1 + 2f^2(s)f'(s)\,\omega_1 \wedge ds + f(s)\tilde{d}\eta \wedge ds, \\ \hat{d}\psi_+(s) &= f^3(s)\,\tilde{d}(\omega_2 \wedge \eta) - 3f^2(s)f'(s)\,\omega_2 \wedge \eta \wedge ds - f^2(s)\,\tilde{d}\omega_3 \wedge ds, \\ \hat{d}\psi_-(s) &= f^3(s)\,\tilde{d}(\omega_3 \wedge \eta) - 3f^2(s)f'(s)\,\omega_3 \wedge \eta \wedge ds + f^2(s)\,\tilde{d}\omega_2 \wedge ds. \end{aligned} \tag{5}$$

As a consequence of the previous considerations can be obtained the following result.

Proposition 2.1. *Let* $(L, \eta, \omega_1, \omega_2, \omega_3)$ *be a* SU(2) *manifold such that is contact metric, that is*

$$\tilde{d}\eta = -2\omega_1,$$

whose underlying metric g_L *is Einstein with positive scalar curvature. Then, the* SU(3)*-structure* $(\omega(s), \psi_+(s), \psi_-(s))$ *on* $N = \mathbb{R}^+ \times L$ *defined by*

$$\begin{aligned} \omega(s) &= s^2\omega_1 + s\,\eta \wedge ds, \\ \psi_+(s) &= s^3\omega_2 \wedge \eta - s^2\omega_3 \wedge ds, \\ \psi_-(s) &= s^3\omega_3 \wedge \eta + s^2\omega_2 \wedge ds, \end{aligned} \tag{6}$$

is almost Kähler and induces the Ricci-flat metric given by the cone metric of g_L*, that is*

$$g_N = ds^2 + s^2 g_L.$$

Proof. Suppose that $(\eta, \omega_1, \omega_2, \omega_3)$ is a SU(2)-structure on L. Then, from (3) and (4) the SU(3)-structure on $I \times L$ defined by (6) is such that its natural metric g_N is exactly the cone metric

$$g_N = ds^2 + s^2 g_L.$$

Finally, considering (5), with $f(s) = s$, and taking into account that the SU(2)-structure satisfies

$$\tilde{d}\eta = -2\omega_1$$

is obtained that the Kähler form of the SU(3)-structure $\omega(s)$ is closed.

□

For the particular case of $(L, \eta, \omega_1, \omega_2, \omega_3)$ being Sasaki-Einstein the following well known result holds.

Proposition 2.2 ([17])**.** *The cone metric over a Riemannian 5-manifold* L *has holonomy contained in* SU(3) *if and only* L *is a Sasaki-Einstein manifold.*

Proof. From (3) and (4) we have that the metric induced by $(\omega(s), \psi_+(s), \psi_-(s))$, described in (6), is exactly the cone metric

$$g_N = ds^2 + s^2 g_L.$$

Now from Table 1 and the fact that g_L is Einstein with positive Einstein constant we conclude that g_N is Ricci-flat.

Finally, considering (5), particularized for $f(s) = s$, can be checked that the condition of the SU(3)-structure to have holonomy on SU(3), which can also be characterized by

$$\hat{d}\omega(s) = 0, \quad \hat{d}\psi_+(s) = 0, \quad \text{and} \quad \hat{d}\psi_-(s) = 0$$

is equivalent to the Sasaki-Einstein condition on the SU(2)-structure, that is

$$\tilde{d}\eta = -2\omega_1, \quad \tilde{d}\omega_3 = -3\eta \wedge \omega_2, \quad \text{and} \quad \tilde{d}\omega_2 = 3\eta \wedge \omega_3.$$

□

Consider now a family of SU(2)-structures $(\eta(s), \omega_1(s), \omega_2(s), \omega_3(s))$ on L, defined for any s in some open interval I, and such that

$$\boxed{\begin{aligned} \eta(s) &= f(s)\,\eta, \\ \omega_1(s) &= f^2(s)\,(\cos s\,\omega_1 + \sin s\,\omega_3), \\ \omega_2(s) &= -f^2(s)\,\omega_2, \\ \omega_3(s) &= f^2(s)\,(\sin s\,\omega_1 - \cos s\,\omega_3), \end{aligned}}$$

where $f = f(s) > 0$ is a real differentiable function on I. Again, as a consequence of (2), the differential forms $(\omega(s), \psi_+(s), \psi_-(s))$ given by

$$\begin{aligned} \omega(s) &= \omega_1(s) + \eta(s) \wedge ds, \\ \psi_+(s) &= \omega_2(s) \wedge \eta(s) - \omega_3(s) \wedge ds, \\ \psi_-(s) &= \omega_3(s) \wedge \eta(s) + \omega_2(s) \wedge ds, \end{aligned}$$

describe a SU(3)-structure on $I \times L$ for any $s \in I$. Now the forms $\omega(s), \psi_+(s)$ and $\psi_-(s)$ can be expressed in terms of the function $f(s)$ and the SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on L, as follows

$$\begin{aligned} \omega(s) &= f^2(s)(\cos s\,\omega_1 + \sin s\,\omega_3) + f(s)\eta \wedge ds, \\ \psi_+(s) &= -f^3(s)\omega_2 \wedge \eta - f^2(s)(\sin s\,\omega_1 - \cos s\,\omega_3) \wedge ds, \\ \psi_-(s) &= f^3(s)(\sin s\,\omega_1 - \cos s\,\omega_3) \wedge \eta - f^2(s)\omega_2 \wedge ds. \end{aligned} \tag{7}$$

Thus, can be checked that the metric induced by the SU(3)-structure, namely g_N is exactly the warped product metric

$$g_N = ds^2 + f^2(s)\,g_L \tag{8}$$

on $I \times_f L$, where g_L denotes the metric induced by the SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on L.

From (7) we have that

$$\begin{aligned} \hat{d}\omega(s) =& f^2(s)(\cos s\,\tilde{d}\,\omega_1 + \sin s\,\tilde{d}\,\omega_3) + 2f(s)f'(s)\,(\cos s\,\omega_1 + \sin s\,\omega_3) \wedge ds \\ &+ f^2(s)(-\sin s\,\omega_1 + \cos s\,\omega_3) \wedge ds + f(s)\tilde{d}\eta \wedge ds, \\ \hat{d}\psi_+(s) =& - f^3(s)\,\tilde{d}(\omega_2 \wedge \eta) + 3f^2(s)f'(s)\,\omega_2 \wedge \eta \wedge ds \\ &- f^2(s)\,(\sin s\,\tilde{d}\,\omega_1 - \cos s\,\tilde{d}\omega_3) \wedge ds, \\ \hat{d}\psi_-(s) =& f^3(s)\,(\sin s\,\tilde{d}(\omega_1 \wedge \eta) - \cos s\,\tilde{d}(\omega_3 \wedge \eta)) \\ &- 3f^2(s)f'(s)\,(\sin s\,\omega_1 - \cos s\,\omega_3) \wedge \eta \wedge ds \\ &- f^3(t)(\cos s\,\omega_1 + \sin s\,\omega_3) \wedge \eta \wedge ds - f^2(s)\,\tilde{d}\omega_2 \wedge ds, \end{aligned} \tag{9}$$

where we use the previously fixed notation on the differentials.

From all the previous considerations can be obtained the following result.

Proposition 2.3. *Let $(L, \eta, \omega_1, \omega_2, \omega_3)$ be a double hypo structure, that is*

$$\tilde{d}\eta = -2\omega_1, \quad \textit{and} \quad \tilde{d}\omega_3 = -3\eta \wedge \omega_2,$$

and such that its underlying metric g_L is Einstein with positive Einstein constant. Then the SU(3)*-structure* $(\omega(s), \psi_+(s), \psi_-(s))$ *on* $N = (0, \pi) \times L$ *defined by*

$$\begin{aligned} \omega(s) &= \sin^2 s(\cos s\,\omega_1 + \sin s\,\omega_3) \sin s\,\eta \wedge ds, \\ \psi_+(s) &= -\sin^3 s\,\omega_2 \wedge \eta - \sin^2 s(\sin s\,\omega_1 - \cos s\,\omega_3) \wedge ds, \\ \psi_-(s) &= \sin^3 s(\sin s\,\omega_1 - \cos s\,\omega_3) \wedge \eta - \sin^2 s\,\omega_2 \wedge ds \end{aligned} \tag{10}$$

is a half-flat SU(3)*-structure on* $(0, \pi) \times L$ *inducing the Einstein* sin*-cone metric of* g_L*, that is*

$$g_N = ds^2 + \sin^2 s g_L.$$

Proof. From (7) and (8) we have that the metric induced by the SU(3)-structure described by (10) is exactly the sin-cone metric

$$g_N = ds^2 + \sin^2 s\, g_L.$$

Thus, from Table 1 and the fact that g_L is Einstein with positive Einstein constant, we can assert that g_N is Einstein.

Finally from (9) and taking into account that the SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ is double hypo we have that

$$\begin{aligned} \hat{d}(\omega(s) \wedge \omega(s)) =& \hat{d}(\sin^4 s\,\omega_1 \wedge \omega_1 + 2\sin^3 s \cos s\,\omega_1 \wedge \eta \wedge ds \\ & + 2\sin^4 s\,\omega_3 \wedge \eta \wedge ds) = 4\sin^3 s \cos s\,\omega_1 \wedge \omega_1 \wedge ds \\ & - 4\sin^3 s\, \cos s\,\omega_1 \wedge \omega_1 \wedge ds = 0, \\ \hat{d}\psi_+(s) =& -\sin^3 s\,\tilde{d}(\omega_2 \wedge \eta) + 3\sin^2 s\, \cos s\,\omega_2 \wedge \eta \wedge ds \\ & - \sin^2 s\,(\sin s\,\tilde{d}\omega_1 - \cos s\,\tilde{d}\omega_3) \wedge ds = 0. \end{aligned}$$

Therefore, $(\omega(s), \psi_+(s), \psi_+(s))$ is half-flat. □

For the particular case of $(L, \eta, \omega_1, \omega_2, \omega_3)$ being Sasaki-Einstein the following well known result holds.

Theorem 2.4 ([17])**.** *Let* $(L, \eta, \omega_1, \omega_2, \omega_3)$ *be a Sasaki-Einstein manifold, that is*

$$\tilde{d}\eta = -2\omega_1, \quad \tilde{d}\omega_3 = -3\eta \wedge \omega_2 \quad \textit{and} \quad \tilde{d}\omega_2 = 3\eta \wedge \omega_3.$$

The SU(3)*-structure* $(\omega(s), \psi_+(s), \psi_-(s))$ *on* $N = (0, \pi) \times L$ *defined by* (10) *is a nearly Kähler* SU(3)*-structure on* $(0, \pi) \times L$ *inducing the Einstein* sin*-cone metric of* g_L*, that is*

$$g_N = ds^2 + \sin^2 s g_L.$$

Proof. From (7) and (8) we have that the metric induced by the SU(3)-structure described by (10) is exactly the sin-cone metric

$$g_N = ds^2 + \sin^2 s g_L.$$

Now, from Table 1 and the fact that the metric g_L, induced by a Sasaki-Einstein structure is Einstein with positive scalar curvature we conclude that g_N is Einstein.

Finally, from (9) particularized for $f(s) = \sin s$ and taking into account that the SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ is Sasaki-Einstein we have that

$$\begin{aligned}
\hat{d}\omega(s) =& -3\sin^3 s\,\eta\wedge\omega_2 + 2\sin s\cos s(\cos s\,\omega_1 + \sin s\,\omega_3)\wedge ds\\
&+\sin^2 s(-\sin s\,\omega_1 + \cos s\,\omega_3)\wedge ds - 2\sin s\,\omega_1\wedge ds\\
=& -3\sin^3 s\,\eta\wedge\omega_2 - 3\sin^3 s\,\omega_1\wedge ds + 3\sin^2 s\cos s\,\omega_3\wedge ds\\
=& \,3\psi_+(s),\\
\hat{d}\psi_-(s) =& \sin^4 s\,\tilde{d}(\omega_1\wedge\eta) - \sin^3 s\,\cos s\,\tilde{d}(\omega_3\wedge\eta)\\
&-3\sin^2 s\,\cos s(\sin s\,\omega_1 - \cos s\,\omega_3)\wedge\eta\wedge ds\\
&-\sin^3 s\,\cos s\,\omega_1\wedge\eta\wedge ds - \sin^4 s\,\omega_3\wedge\eta\wedge ds - \sin^2\tilde{d}\omega_2\wedge ds\\
=& -2\sin^4 s\,\omega_1\wedge\omega_1 - 4\sin^3 s\cos s\,\omega_1\wedge\eta\wedge ds - 4\sin^4 s\,\omega_3\wedge\eta\wedge ds\\
=& -2\omega(s)\wedge\omega(s),
\end{aligned}$$

and therefore $(\omega(s), \psi_+(s), \psi_-(s))$ is nearly Kähler.

□

Let us consider now a family of SU(2)-structures $(\eta(s), \omega_1(s), \omega_2(s), \omega_3(s))$ on L defined for any s in some interval I, and such that

$$\boxed{\begin{aligned}
\eta(s) &= f(s)\,\eta,\\
\omega_1(s) &= f^2(s)\,(\cos s\,\omega_1 - \sin s\,\omega_2),\\
\omega_2(s) &= -f^2(s)\,\omega_3,\\
\omega_3(s) &= f^2(s)\,(\sin s\,\omega_1 + \cos s\,\omega_2),
\end{aligned}}$$

where $f = f(t) > 0$ is a real differentiable function on I. As a consequence of (2), the differential forms $(\omega(s), \psi_+(s), \psi_-(s))$ given by

$$\begin{aligned}
\omega(s) &= \omega_1(s) + \eta(s)\wedge ds,\\
\psi_+(s) &= \omega_2(s)\wedge\eta(s) - \omega_3(s)\wedge ds,\\
\psi_-(s) &= \omega_3(s)\wedge\eta(s) + \omega_2(s)\wedge ds,
\end{aligned}$$

describe an SU(3)-structure on $I\times L$ for any $s\in I$. Now the forms $\omega(s), \psi_+(s)$ and $\psi_-(s)$ can be expressed in terms of the function $f(s)$ and the SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on L, as follows

$$\begin{aligned}
\omega(s) &= f^2(s)(\cos s\,\omega_1 - \sin s\,\omega_2) + f(s)\eta\wedge ds,\\
\psi_+(s) &= -f^3(s)\omega_3\wedge\eta - f^2(s)(\sin s\,\omega_1 + \cos s\,\omega_2)\wedge ds,\\
\psi_-(s) &= f^3(s)(\sin s\,\omega_1 + \cos s\,\omega_2)\wedge\eta - f^2(s)\omega_3\wedge ds,
\end{aligned}\tag{11}$$

Thus, the metric induced by the SU(3)-structure, namely g_N is exactly the warped product metric

$$g_N = ds^2 + f^2(s)\,g_L \tag{12}$$

on $I \times_f L$, where g_L denotes the metric induced by the SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ on L.

From (11) we have that

$$
\begin{aligned}
\hat{d}\omega(s) =& f^2(s)(\cos s\, \tilde{d}\,\omega_1 - \sin s\, \tilde{d}\,\omega_2) + 2f(s)f'(s)\,(\cos s\,\omega_1 - \sin s\,\omega_2) \wedge ds \\
&+ f^2(s)(-\sin s\,\omega_1 - \cos s\,\omega_2) \wedge ds + f(s)\tilde{d}\eta \wedge ds, \\
\hat{d}\psi_+(s) =& - f^3(s)\,\tilde{d}(\omega_3 \wedge \eta) + 3f^2(s)f'(s)\,\omega_3 \wedge \eta \wedge ds \\
&- f^2(s)\,(\sin s\,\tilde{d}\,\omega_1 + \cos s\,\tilde{d}\omega_2) \wedge ds, \\
\hat{d}\psi_-(s) =& f^3(s)\,(\sin s\,\tilde{d}(\omega_1 \wedge \eta) + \cos s\,\tilde{d}(\omega_2 \wedge \eta)) \\
&- 3f^2(s)f'(s)\,(\sin s\,\omega_1 + \cos s\,\omega_2) \wedge \eta \wedge ds \\
&- f^3(s)(\cos s\,\omega_1 - \sin s\,\omega_2) \wedge \eta \wedge ds - f^2(s)\,\tilde{d}\omega_3 \wedge ds,
\end{aligned}
\tag{13}
$$

where we use the previously fixed notation on the differentials.

For the particular case of $(\eta, \omega_1, \omega_2, \omega_3)$ being a nearly hypo Einstein structure the following result holds.

Proposition 2.5. *Let $(L, \eta, \omega_1, \omega_2, \omega_3)$ be a nearly hypo structure, that is*

$$\tilde{d}\omega_3 = -3\eta \wedge \omega_2, \quad \textit{and} \quad \tilde{d}(\eta \wedge \omega_1) = -2\omega_3 \wedge \omega_3,$$

which underlying metric g_L is Einstein with positive Einstein constant. Thus, the SU(3)*-structure $(\omega(s), \psi_+(s), \psi_-(s))$ on $N = (0, \pi) \times L$ defined by*

$$
\begin{aligned}
\omega(s) &= \sin^2 s(\cos s\,\omega_1 - \sin s\,\omega_2)\sin s\,\eta \wedge ds, \\
\psi_+(s) &= -\sin^3 s\,\omega_3 \wedge \eta - \sin^2 s(\sin s\,\omega_1 + \cos s\,\omega_2) \wedge ds, \\
\psi_-(s) &= \sin^3 s(\sin s\,\omega_1 + \cos s\,\omega_2) \wedge \eta - \sin^2 s\,\omega_3 \wedge ds
\end{aligned}
\tag{14}
$$

is a nearly half-flat SU(3)*-structure on $(0, \pi) \times L$ inducing the Einstein* sin*-cone metric of g_L, that is*

$$g_N = ds^2 + \sin^2 s\, g_L.$$

Proof. From (11) and (12) we have that the metric induced by the SU(3)-structure described in (14) is exactly the sin-cone metric of g_L.
Thus, from Table 1 and the fact that the metric g_L, induced by a Sasaki-Einstein structure is Einstein with positive scalar curvature we conclude that g_N is Einstein.

Finally, from (13) and taking into account that the SU(2)-structure $(\eta, \omega_1, \omega_2, \omega_3)$ is Sasaki-Einstein we have that

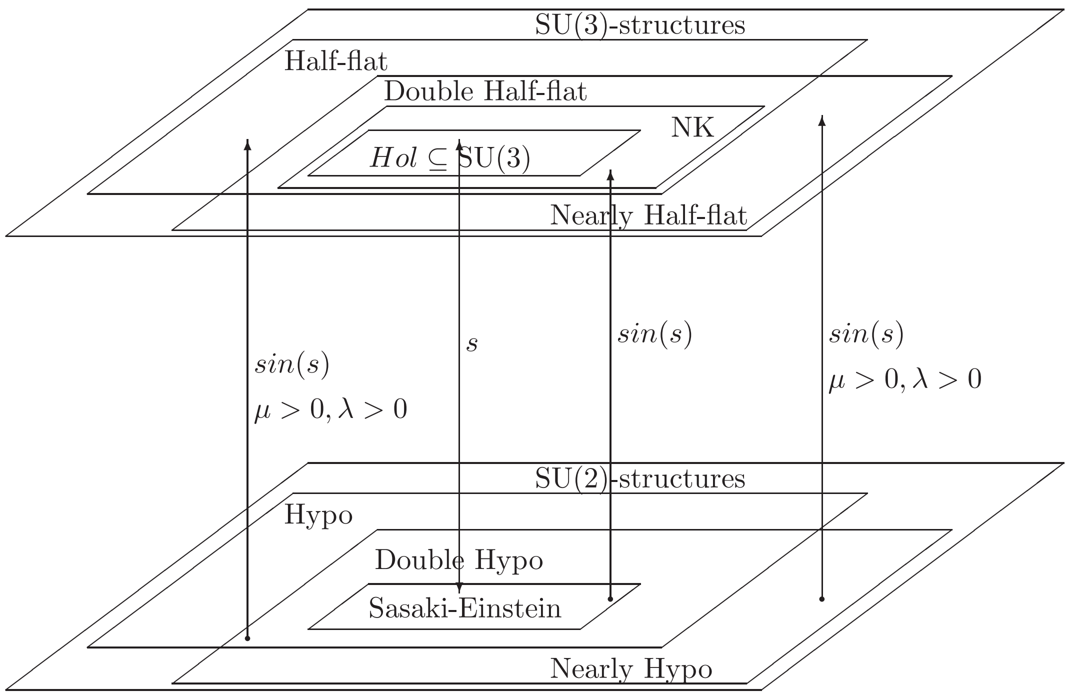
$$
\begin{aligned}
\hat{d}\psi_-(s) =& - 2\sin^4 s\,\omega_3 \wedge \omega_3 - 3\sin^3 s\cos s\,\omega_1 \wedge \eta \wedge ds \\
&- 3\sin^2 s\cos^2 s\,\omega_2 \wedge \eta \wedge ds - \sin^3 s\cos s\,\omega_1 \wedge \eta \wedge ds \\
&+ \sin^4 s\,\omega_2 \wedge \eta \wedge ds + 3\sin^2 s\,\eta \wedge \omega_2 \wedge ds \\
=& - 2\sin^4 s\,\omega_3 \wedge \omega_3 - 4\sin^3 s\cos s\,\omega_1 \wedge \eta \wedge ds \\
&+ 4\sin^4 s\,\omega_2 \wedge \eta \wedge ds = -2\omega(s) \wedge \omega(s).
\end{aligned}
$$

Therefore $(\omega(s), \psi_+(s), \psi_-(s))$ is nearly half-flat.

□

In order to summarize some known results about the warped products of SU(2)-structures we give the following diagram:

Warped product of Einstein SU(2)-structures



Remark 2.6. *The notation of μ and λ corresponds with the scalar curvature of the base and the total manifold respectively.*

3. Warped product of SU(3)-structures

This section is devoted to the construction of Einstein G$_2$-structures on $I \times N$ from Einstein SU(3)-structures on N.

It is well known that SU(3)-structures and G$_2$-structures are closely related. In fact, if (N, ω, ψ_+) is a 6-dimensional manifold endowed with a SU(3)-structure then the 3-form

$$\varphi = \omega \wedge dt + \psi_+,$$

defines a G$_2$-structure on the 7-dimensional manifold $N \times \mathbb{R}$ where t denotes the coordinate in $\mathbb{R}$.

Regarding the converse, Cabrera in [7] shows that if N is an orientable hypersurface of a G$_2$ manifold (M, φ) then the pair of forms on N defined by

$$\omega = \iota_U \varphi \quad \textit{and} \quad \psi_+ = \pi^* \varphi,$$

with U the unitary vector field of M normal to N and π the projection of M onto N, describe a SU(3)-structure on N.

However, we want to construct G_2-structures from SU(3)-structures in such a way that the Einstein condition of the induced metric is preserved.

Let $(N, \omega, \psi_+, \psi_-)$ be a SU(3) manifold. Consider a family of SU(3)-structures $(\omega(t), \psi_+(t), \psi_-(t))$ on N, for any t in some interval I, such that

$$\boxed{\begin{aligned} \omega(t) &= f^2(t)\omega, \\ \psi_+(t) &= f^3(t)\psi_+, \\ \psi_-(t) &= f^3(t)\psi_-, \end{aligned}}$$

where $f = f(t) > 0$ is a real differentiable function on I. For any $t \in I$, the 3-form $\varphi(t)$ on $N \times I$ given by

$$\varphi(t) = \omega(t) \wedge dt + \psi_+(t)$$

is a G_2 form.

In these conditions, the forms $\varphi(t)$ and $*_{\varphi(t)}\varphi(t)$ can be expressed, in terms of the function $f(t)$ and the SU(3)-structure (ω, ψ_+, ψ_-) on N, as follows

$$\begin{aligned} \varphi(t) &= f^2(t)\omega \wedge dt + f^3(t)\psi_+, \\ *_{\varphi(t)}\varphi(t) &= \frac{1}{2} f^4(t)\omega \wedge \omega + f^3(t)\psi_- \wedge dt. \end{aligned} \tag{15}$$

Thus, can be checked that the induced metric, namely g_M is exactly the warped product metric

$$g_M = dt^2 + f^2(t) g_N \tag{16}$$

on $M = I \times_f N$, where g_N denotes the metric induced by the SU(3)-structure (ω, ψ_+, ψ_-) on N.

If we denote by d the differential on $M = I \times_f N$ and by $\hat{d}$ the differential on N from (15) we have that

$$\begin{aligned} d\varphi(t) &= f^2(t)\hat{d}\omega \wedge dt - 3f^2(t)f'(t)\psi_+ \wedge dt + f^3(t)\hat{d}\psi_+, \\ d(*_{\varphi(t)}\varphi(t)) &= f^4(t)\hat{d}\omega \wedge \omega + 2f^3(t)f'(t)\omega \wedge \omega \wedge dt + f^3(t)\hat{d}\psi_- \wedge dt. \end{aligned} \tag{17}$$

As a consequence of the previous considerations can be obtained the following result.

Proposition 3.1. *Let $(N, \omega, \psi_+, \psi_-)$ be a nearly half-flat structure, that is*

$$\hat{d}\psi_- = -2\omega \wedge \omega.$$

If its underlying metric g_N is Einstein with positive Einstein constant thus the G_2-structure $\varphi(t)$ on $M = \mathbb{R}^+ \times N$ defined by

$$\begin{aligned} \varphi(t) &= t^2\omega \wedge dt + t^3\psi_+, \\ *_{\varphi(t)}\varphi(t) &= \frac{1}{2} t^4\omega \wedge \omega + t^3\psi_- \wedge dt. \end{aligned} \tag{18}$$

is a coclosed G_2 form on M inducing the Ricci-flat metric given by the cone metric of g_N, that is

$$g_M = dt^2 + t^2 g_N.$$

Proof. From (15) and (16) we have that the metric induced by (18) is exactly the cone metric

$$g_M = dt^2 + t^2 g_N.$$

Now from Table 1 and the fact that g_N is Einstein with positive Einstein constant we conclude that g_M is Ricci-flat.

On the other hand, from (17) particularized for $f(t) = t$ and taking into account that the SU(3)-structure is nearly half-flat is obtained that $\varphi(t)$ is coclosed.

□

For the particular case of $(N, \omega, \psi_+, \psi_-)$ being nearly Kähler the following well known result is obtained.

Proposition 3.2. [5, 6] *The cone metric over a Riemannian 6-manifold N has holonomy contained in* G_2 *if and only if N is a nearly Kähler manifold.*

Proof. Suppose that (ω, ψ_+, ψ_-) is an SU(3)-structure on N. Then, from (15) and (16) the G_2-structure on $I \times N$ defined by (18) is such that its induced metric g_M is exactly the cone metric

$$g_M = dt^2 + t^2 g_N.$$

Finally from (17) can be checked that the condition of the G_2-structure to have holonomy contained in G_2, which can also be characterized by

$$d\varphi(t) = 0 \quad \textit{and} \quad d *_{\varphi(t)} \varphi(t) = 0$$

is equivalent to the nearly Kähler condition on the SU(3)-structure that is

$$\hat{d}\omega = 3\psi_+ \quad \textit{and} \quad \hat{d}\psi_- = -2\omega \wedge \omega.$$

□

Let us consider now a family of SU(3)-structures $(\omega(t), \psi_+(t), \psi_-(t))$ on N, for any t in some interval I, such that

$$\boxed{\begin{aligned} \omega(t) &= f^2(t)\omega, \\ \psi_+(t) &= f^3(t)(\cos t\, \psi_+ - \sin t\, \psi_-), \\ \psi_-(t) &= f^3(t)(\sin t\, \psi_+ + \cos t\, \psi_-), \end{aligned}}$$

where $f = f(t) > 0$ is a real differentiable function on I. For any $t \in I$, the 3-form $\varphi(t)$ on $N \times I$ given by

$$\varphi(t) = \omega(t) \wedge dt + \psi_+(t)$$

is a G_2 form.

In these conditions, the forms $\varphi(t)$ and $*_{\varphi(t)}\varphi(t)$ can be described in terms of the function $f(t)$ and the SU(3)-structure (ω, ψ_+, ψ_-) on N, as follows

$$\begin{aligned} \varphi(t) &= f^2(t)\omega \wedge dt + f^3(t)(\cos t\psi_+ - \sin t\psi_-), \\ *_{\varphi(t)}\varphi(t) &= \frac{1}{2} f^4(t)\omega \wedge \omega + f^3(t)(\sin t\psi_+ + \cos t\psi_-) \wedge dt. \end{aligned} \tag{19}$$

Thus, the induced metric is exactly the warped product metric

$$g_M = dt^2 + f^2(t) g_N, \tag{20}$$

on $I \times_f N$, where g_N denotes the metric induced by the SU(3)-structure (ω, ψ_+, ψ_-) on N^6.

From (19) we have that

$$\begin{aligned} d\varphi(t) =& f^2(t)\hat{d}\omega \wedge dt - 3f^2(t)f'(t)(\cos t\psi_+ \wedge dt - \sin t\psi_- \wedge dt) \\ & + f^3(t)\sin t\psi_+ \wedge dt + f^3(t)\cos t\psi_- \wedge dt, \\ & f^3(t)\cos t\hat{d}\psi_+ - f^3(t)\sin t\hat{d}\psi_-, \\ d(*_{\varphi(t)}\varphi(t)) =& f^4(t)\hat{d}\omega \wedge \omega + 2f^3(t)f'(t)\omega \wedge \omega \wedge dt \\ & + f^3(t)\sin t\hat{d}\psi_+ \wedge dt + f^3(t)\cos t\hat{d}\psi_- \wedge dt, \end{aligned} \tag{21}$$

where we adopt the previously mentioned notation on the differentials.

Proposition 3.3. *Let $(N, \omega, \psi_+, \psi_-)$ be a double half-flat structure, that is*

$$\hat{d}\psi_+ = 0, \quad \textit{and} \quad \hat{d}\psi_- = -2\omega \wedge \omega.$$

If its underlying metric g_N is Einstein with positive Einstein constant thus the G_2-structure $\varphi(t)$ on $M = (0, \pi) \times N$ defined by

$$\begin{aligned} \varphi(t) &= \sin^2 t\omega \wedge dt + \sin^3 t(\cos t\psi_+ - \sin t\psi_-), \\ *_{\varphi(t)}\varphi(t) &= \frac{1}{2}\sin^4 t\omega \wedge \omega + \sin^3 t(\sin t\psi_+ + \cos t\psi_-) \wedge dt \end{aligned} \tag{22}$$

is a coclosed G_2 form on $M = (0, \pi) \times N$ inducing the Einstein sin-*cone metric of g_N.*

Proof. From (19) and (20) we have that the metric induced by

$$\varphi(t) = \sin^2 t\omega \wedge dt + \sin^3 t(\cos t\psi_+ - \sin t\psi_-),$$

is exactly the sin-cone metric

$$g_M = dt^2 + \sin^2 t g_N.$$

Now, Table 1 and the fact that g_N is Einstein with positive Einstein constant let us conclude that g_M is Einstein.

Finally, from (21) particularized for $f(t) = \sin t$ and the fact that the SU(3)-structure is double half-flat is obtained that

$$\begin{aligned} d *_{\varphi(t)} \varphi(t) =& \sin^4 t\hat{d}\omega \wedge \omega + 2\sin^3 t\cos t\omega \wedge \omega \wedge dt + \sin^4 t\hat{d}\psi_+ \wedge dt \\ & + \sin^3 t\cos t\hat{d}\psi_- \wedge dt = 0, \end{aligned}$$

and therefore $\varphi(t)$ is coclosed. □

If, in particular, $(N, \omega, \psi_+, \psi_-)$ is a nearly Kähler manifold the following result holds.

Theorem 3.4. [17] *Let (ω, ψ_+, ψ_-) be a nearly Kähler structure, that is*

$$\hat{d}\omega = 3\psi_+ \quad \text{and} \quad \hat{d}\psi_- = -2\omega \wedge \omega.$$

The G_2-structure on $M = (0, \pi) \times N$ defined by (22) is a nearly parallel G_2 form on $(0, \pi) \times N$ inducing the Einstein sin*-cone metric of g_N,*

$$g_M = dt^2 + \sin^2 t g_N.$$

Proof. From (19) and (20) we have that the metric induced by (22) is exactly the sin-cone metric. Now, from Table 1 and the fact that the metric induced by a nearly Kähler structure is Einstein with positive Einstein constant we conclude that g_M is Einstein.

Finally, from (21) and taking into account that the SU(3)-structure (ω, ψ_+, ψ_-) is nearly Kähler we have that

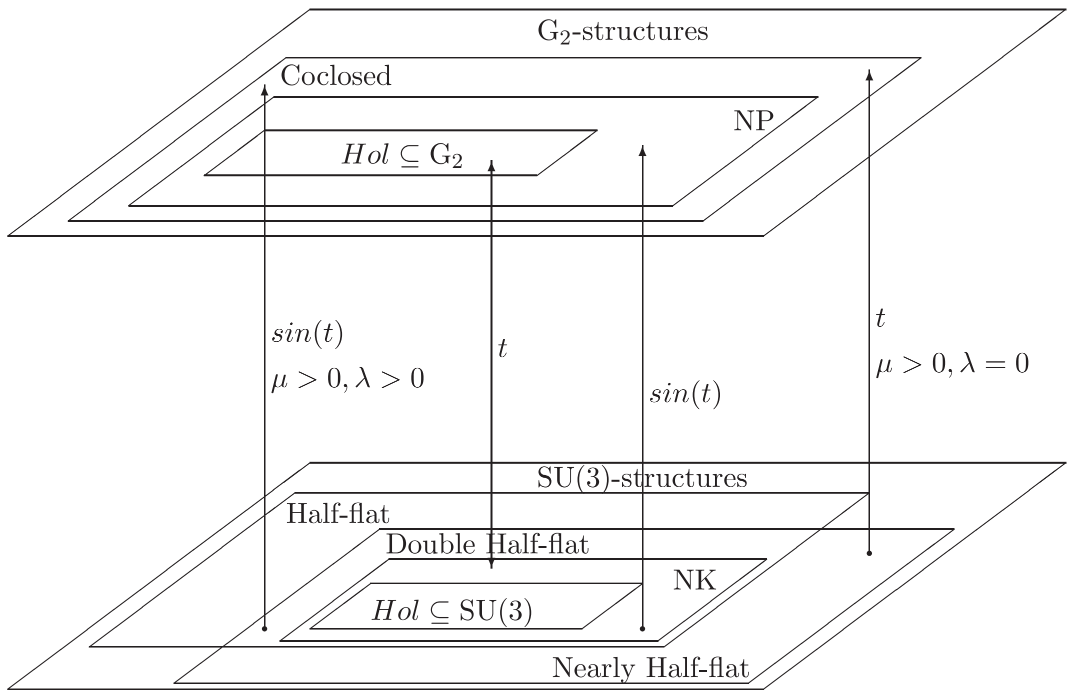
$$\begin{aligned} d\varphi(t) =& 4\sin^4 t\psi_+ \wedge dt + 4\sin^3 t\cos t\psi_- \wedge dt + 2\sin^4 t\omega \wedge \omega \\ =& 4 *_{\varphi(t)} \varphi(t), \end{aligned}$$

and therefore $\varphi(t)$ is nearly Parallel.

□

In order to summarize all the known results about the warped products of SU(3)-structures we give the following diagram:

Warped product of Einstein SU(3)**-structures**



Remark 3.5. *The notation of μ and λ corresponds with the scalar curvature of the base and the total manifold respectively.*

4. The Einstein coclosed G$_2$-structure on $(0, \pi) \times S^3 \times S^3$

Next, as a consequence of Proposition 3.3 we construct a 7-dimensional (non-compact) manifold M endowed with a coclosed G$_2$ form φ whose induced metric is Einstein. We also show that φ does not define a nearly parallel G$_2$-structure neither a 3-Sasakian.

Example 4.1. *Let us consider the sphere S^3, viewed as the Lie group* SU(2)*, with the basis of left invariant 1-forms $\{e^1, e^2, e^3\}$ satisfying*

$$de^1 = e^{23}, \quad de^2 = -e^{13}, \quad \text{and} \quad de^3 = e^{12}.$$

Hence, the Lie algebra of $S^3 \times S^3$ is $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and its structure equations are

$$\mathfrak{g} = (e^{23}, -e^{13}, e^{12}, f^{23}, -f^{13}, f^{12}),$$

where $\{f^i\}$ denotes the basis of 1-forms on the second sphere.

Now, we consider the basis $\{h^1,\dots,h^6\}$ of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ given by

$$h^1=\frac{1}{4}(e^1+f^1),\ h^2=\frac{1}{4}(-e^1+f^1),\ h^3=\frac{\sqrt{2}}{4}e^2,\ h^4=\frac{\sqrt{2}}{4}f^2,\ h^5=\frac{\sqrt{2}}{4}e^3,\ h^6=\frac{\sqrt{2}}{4}f^3.$$

With respect to this basis, the structure equations of the Lie algebra $\mathfrak{g}$ of $S^3\times S^3$ turn into

$$\mathfrak{g}=(2h^{35}+2h^{46},-2h^{35}+2h^{46},-2h^{15}+2h^{25},-2h^{16}-2h^{26},2h^{13}-2h^{23},2h^{14}+2h^{24}).$$

We define the SU(3)*-structure (ω,ψ_+,ψ_-) on $S^3\times S^3$ by*

$$\begin{aligned}\omega &= h^{12}+h^{34}+h^{56},\\ \psi_+ &= h^{135}-h^{146}-h^{236}-h^{245},\\ \psi_- &= -h^{246}+h^{235}+h^{145}+h^{136}.\end{aligned}\tag{23}$$

Then, an easy calculation shows that

$$\begin{aligned}\widehat{d}\omega &= 3\psi_+ + \nu_3,\\ \widehat{d}\psi_+ &= 0,\\ \widehat{d}\psi_- &= -2\omega\wedge\omega,\end{aligned}\tag{24}$$

which means that (ω,ψ_+,ψ_-) is a double half-flat structure (in the sense of Proposition 3.3) on $S^3\times S^3$, where $\nu_3\in\Omega^3_{12}(S^3\times S^3)$ is given by

$$\nu_3=-h^{135}+h^{146}-h^{236}-h^{245}+2h^{235}+2h^{246}.\tag{25}$$

The second and third equations of (24) *imply that the* SU(3)*-structure defined by* (23) *is double half-flat but it is not nearly Kähler because $\nu_3\neq 0$, and hence $\widehat{d}\omega\neq 3\psi_+$. We have that the metric g induced by $(\omega,\Psi=\psi_++i\psi_-)$ on $S^3\times S^3$ is Einstein with positive scalar curvature. Indeed, g is given by*

$$g=(h^1)^2+(h^2)^2+(h^3)^2+(h^4)^2+(h^5)^2+(h^6)^2,\tag{26}$$

and its Ricci curvature tensor is

$$Ric=4\,(h^1\otimes h^1+h^2\otimes h^2+h^3\otimes h^3+h^4\otimes h^4+h^5\otimes h^5+h^6\otimes h^6).$$

Thus, g is Einstein with Einstein constant $\mu=4$.

Now, by Proposition 3.3, we know that the 3-form φ defined by

$$\begin{aligned}\varphi=&\sin^2 t\,(h^{12}+h^{34}+h^{56})\wedge dt+\sin^3 t\,\cos t\,(h^{135}-h^{146}-h^{236}-h^{245})\\ &-\sin^4 t\,(h^{136}+h^{145}+h^{235}-h^{246})\end{aligned}$$

is a coclosed G_2 *form on $M=(0,\pi)\times S^3\times S^3$. Moreover, we obtain that*

$$\begin{aligned}d\varphi &= -4*_\varphi\varphi(t)+\ sin^2(t)\nu_3\wedge dt,\\ d\big(*_\varphi\varphi\big) &= 0,\end{aligned}$$

where ν_3 is defined by (25)*. These equations not only imply that φ is coclosed but also imply that the* G_2*-structure defined by φ is not nearly-parallel since $\nu_3\wedge dt\neq 0$. Also, by Proposition 3.3, we know that φ induces the Einstein metric*

$$g_\varphi=dt^2+\sin^2 t\,g$$

where g is the Einstein metric on $S^3 \times S^3$ given by (26)*. Now, a direct calculation shows that the Einstein constant of g_φ is $\lambda = \frac{24}{5}$. Thus, the metric g_φ is not 3-Sasakian since, the Einstein constant of a 3-Sasakian metric on a 7-dimensional manifold is* 6*.*

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